International Journal of Theoretical Physics, Vol. 8, No. 4 (1973), pp. 301–306

On the Motion of a Charged Particle with Radiation Reaction

N. D. SEN GUPTA

Tata Institute of Fundamental Research, Bombay—5, India

Received: 9 October 1972

Abstract

The object of this paper is to show that initial acceleration is uniquely determined by the condition that the physical solutions of the Dirac–Lorentz equation are regular at $2e^2/3mc^3 = 0$. The method for obtaining this initial acceleration in the general case is given.

1. *Introduction*

Recently, in a series of papers, the author (Sen Gupta, 1970, 1971, 1972) has studied the problem of relativistic motion of a charge of particle in an external electromagnetic field taking account of the reaction due to radiation. The appearance of the so-called 'unphysical' solution of the Dirac–Lorentz equation of motion is quite well known. Since the appearance of the classic paper by Dirac (1938), various prescriptions have been suggested to reject these solutions. In order to incorporate the force of reaction due to radiation, the order of the differential equation of motion is increased by one, so that the initial position and velocity are no longer sufficient to uniquely determine the motion; in addition, the initial acceleration should also be prescribed. This is not in accord with Newtonian mechanics. On the other hand, the equation, being of a higher order, admits of a wider class of solutions. Further, since the parameters of the radiation reaction $\varepsilon = 2e^2/3mc^3$, being the coefficient of the third-order derivate, from the theory of the differential equation, it is known that there are two different classes of solutions to the equation, namely solutions which are regular in ε at $\varepsilon = 0$ and which are singular at $\varepsilon = 0$. Since the magnitude of ε, which has the dimension of time, is extremely small in comparison to the order of time involved in the usual physical problems, it is but natural to confine oneself to those solutions which are regular in ε at $\varepsilon = 0$ so that they may be bounded. The object of this paper is to show that, in general, this condition, along with the initial velocity and position, uniquely determines the motion. In other words, for the given initial velocity and position the regularity of the solution with respect to ε uniquely determines the initial acceleration. With this prescription the author was able to discuss the motion in some special cases where one or more exact integrals can be obtained. From the exact integrals one can easily determine the constants of integration for the solution to be regular in ε. However, in most cases the exact integrals may not be obtained, then the question arises, what should be the unique initial acceleration which will lead to the desired solution? Thus by this restriction we are still in the realm of Newtonian mechanics, in so far as the motion is uniquely determined by the initial velocity and position.

The next section is devoted to the determination of the initial acceleration and to showing the uniqueness of the solution in any external field. In Section 3 the possible range of variation of the acceleration is obtained. The paper ends with a short discussion.

2. *The Equation of Motion and the Initial Conditions*

The Lorentz–Dirac equation of motion for a charge particle moving in an external electromagnetic field, $f_{\mu\nu}$, with the radiation reaction may be written as

$$\dot{v}_\mu - \varepsilon(\ddot{v}_\mu + v_\mu \dot{v}_\nu \dot{v}^\nu) = \frac{e}{mc} f_{\mu\nu} v^\nu \tag{2.1}$$

A dot denotes differentiation with respect to the proper time τ;

$$v_\mu = \frac{\dot{x}_\mu}{c}, \qquad v_\mu v^\mu = 1, \qquad \varepsilon = \frac{2e^2}{3mc^3} \tag{2.2}$$

It is well known from the theory of differential equations that the general solutions of equation (2.1), i.e. v_μ and x_μ considered as functions of the parameter ε, have a singularity at $\varepsilon = 0$. This is due to the fact that ε is the coefficient of the highest order derivative and at $\varepsilon = 0$ the order of the equation is reduced. In this section we will discuss how to choose the initial acceleration so as to confine ourselves to the class of solutions which are bounded as $\varepsilon \to 0$, i.e. which are regular in ε. Since equation (2.1) is third order in x_μ with respect to the independent variable τ, the unique solution is obtained only when x_μ, v_μ, and $\dot{v}_\mu$ are given (subject to the restriction in equation (2.2)) at any arbitrary instant, which we may take to be $t = 0$ and $\tau = 0$. Naturally we restrict ourselves to the solutions which are analytic in τ $(0 \leqslant \tau \leqslant \infty)$, this is also demanded from the physical nature of the problem. Hence,

$$x_\mu(\tau) = x_\mu(0) + \tau v_\mu(0) + \frac{\tau^2}{2} \dot{v}_\mu(0) + \frac{\tau^3}{3!} \ddot{v}_\mu(0) + \dots \tag{2.3}$$

$x_\mu(0)$, $v_\mu(0)$ and $\dot{v}_\mu(0)$ being known we can find higher order derivatives at $\tau=0$ from equation (2.1) written in the following form:

$$\ddot{v}_\mu = -v_\mu \dot{v}_\nu \dot{v}^\nu + \frac{1}{\varepsilon}\left(\dot{v}_\mu - \frac{e}{mc} f_{\mu\nu} v^\nu\right) \tag{2.1'}$$

It is clear that any arbitrary choice of $\dot{v}_\mu(0)$ will lead to singular terms in $\ddot{v}_\mu(0)$ at $\varepsilon=0$. If it is possible to find exact integrals, then by inspection one can simply find the initial value $\dot{v}_\mu(0)$ which allows solutions regular in ε. In the absence of exact integrals, let us try to find, in general, the admissible initial value $\dot{v}_\mu(0)$. Let us express

$$\dot{v}_\mu(0) = w_{0,\mu} + \varepsilon w_{1,\mu} + \varepsilon^2 w_{2,\mu} + \dots \tag{2.4}$$

From equation (2.1) one notes that $\ddot{v}_\mu(0)$ will be regular in ε only if

$$w_{0,\mu} = \frac{e}{mc} f_{\mu\nu}(x(0))\, v^\nu(0) \tag{2.5}$$

Thus, $w_{0,\mu}$ is $\dot{v}_\mu(0)$ as obtained from the equation of motion with $\varepsilon=0$. Further, from equation (2.1) it follows that $v_\mu^{(n+2)}(0)$ will not contain terms in $1/\varepsilon$ only if

$$\left[\frac{d^n}{d\tau^n}\left(\dot{v}_\mu - \frac{e}{mc} f_{\mu\nu} v^\nu\right)\right]_{\tau=0} = 0 \qquad \text{as } \varepsilon \to 0 \tag{2.6}$$

for all $n \geqslant 0$. This implies that $w_{0,\mu}$ is nothing but $\dot{v}_{0,\mu}^{(0)}(\tau)$ at $\tau=0$, which satisfies the equation

$$\dot{v}_\mu^{(0)}(\tau) = \frac{e}{mc} f_{\mu\nu}(x(\tau))\, v^{(0)\,\nu}(\tau) \tag{2.7}$$

This is merely the equation of motion, equation (2.1), with $\varepsilon=0$, $\dot{v}_\mu^{(0)}(\tau)$ thus obtained, we can express $w_{1,\mu}, w_{2,\mu}, \dots$ from equation (2.1), in terms of $v_\mu^{(0)}(0)$, $\dot{v}_\mu^{(0)}(0)$, $\ddot{v}_\mu^{(0)}(0), \dots$, e.g.

$$w_{1,\mu} = v_\mu(0)\, w_{0,\nu} w_0{}^\nu + f_\mu^\nu(x(0))\, w_{0,\nu} + \frac{\partial f_\mu^\nu(x(0))}{\partial x_n} v_n(0)\, v_\nu(0) \tag{2.8}$$

and

$$\begin{aligned} w_{2,\mu} = \ddot{v}_\mu^{(0)}(0) + w_{1,\mu} w_{0,\nu} w_0{}^\nu + 2 w_{0,\mu} w_{1,\nu} w_0{}^\nu \\ + f_\mu^\nu(x(0))\, w_{1,\nu} + 2 v_\mu(0)\, w_{1,\nu} w_0{}^\nu \dots \end{aligned} \tag{2.9}$$

The convergence of the series (equation (2.3)) with suitable restriction is well known and is discussed in detail to prove the existence of solutions. We are only restricting to particular initial value. It should be emphasised that equation (2.6) is necessary and sufficient to have a solution which is regular in ε. Since the initial condition $\dot{v}_\mu(0)$ is uniquely determined for given $x_\mu(0)$ and $v_\mu(0)$ we thus arrive at a unique solution of equation (2.1), which remains bounded as $\varepsilon \to 0$. As a matter of fact, it can be directly established that for the given initial $x_\mu(0)$ and $v_\mu(0)$ equation (2.1) has only a unique solution which can be expressed in a power series of ε. Assuming two different power series and substituting them in equation (2.1), it follows that they can differ at best by a constant, which is necessarily zero as they agree at $\tau = 0$. This is inherently due to the fact that equation (2.1) in the limit ε tending to zero is linear in the highest derivative, namely $\dot{v}_\mu$.

In order to emphasise the role of the initial value of $\dot{v}_\mu(0)$ in the nature of the solution, it is worth mentioning the solution for no external field, in which case the exact solution is known (Dirac, 1938; Appendix I). From equation (A.I.3) it follows that the motion is not along a line if $\dot{v}_\mu(0) \neq 0$ or $\mathbf{v}_\mu(0) \neq 0$. In the system in which the particle is initially at rest the motion is along the direction of initial acceleration $\dot{v}_\mu(0)$. The energy increases continuously with time. The solution is singular at $\varepsilon = 0$, excluding the case when $\dot{v}_\mu(0) = 0$. As noted above only in this case is the solution regular in ε. It leads to the physically consistent solution, namely motion with uniform velocity, because in this case, from equations (A.I.1) and (A.I.3), $w = 0$ and $v_\mu(\tau) = v_\mu(0)$.

3. *The Nature of Acceleration in General*

Equation (2.1) can be written as

$$\frac{d}{d\tau}(\sqrt{(-\dot{v}_\mu \dot{v}^\mu)}\, e^{-\tau/\varepsilon}) = \frac{1}{\varepsilon} e^{-\tau/\varepsilon} f_{\mu\nu} v^\nu \frac{\dot{v}^\mu}{\sqrt{(-\dot{v}_\eta \dot{v}^\eta)}} \tag{3.1}$$

$\dot{v}_\mu$ is space-like, so $-\dot{v}_\mu \dot{v}^\mu$ is positive. It is shown in Appendix II that so long as the external field is bounded

$$\left| f_{\mu\nu} v^\nu \frac{\dot{v}^\mu}{\sqrt{(-\dot{v}_\eta \dot{v}_\eta)}} \right| \leqslant M \tag{3.2}$$

M is the largest eigenvalue of $\{f^\nu_\mu\}$. Hence

$$M(e^{-\tau/\varepsilon} - 1) \leqslant \{\sqrt{[-\dot{v}_\mu(\tau)\, \dot{v}^\mu(\tau)]}\, e^{-\tau/\varepsilon} - \sqrt{[-\dot{v}_\mu(0)\, \dot{v}^\mu(0)]}\} \leqslant M(1 - e^{-\tau/\varepsilon}) \tag{3.3}$$

From which it follows that

$$\sqrt{[-\dot{v}_\mu(\tau)\, \dot{v}^\mu(\tau)]} \geqslant (\sqrt{[-\dot{v}_\mu(0)\, \dot{v}^\mu(0)]} - M)\, e^{\tau/\varepsilon} + M \tag{3.4}$$

Thus, for any arbitrary value of $\dot{v}_\mu(0)$, the acceleration will increase more rapidly than $\exp \tau/\varepsilon$ and, further, this limit increases enormously for $\varepsilon \to 0$ even in a very small interval of time. The other bound is given

$$\sqrt{[-\dot{v}_\mu(\tau)\, \dot{v}^\mu(\tau)]} \leqslant (\sqrt{[-\dot{v}_\mu(0)\, \dot{v}^\mu(0)]} + M)\, e^{\tau/\varepsilon} + M \tag{3.5}$$

This shows that the initial acceleration cannot be chosen arbitrarily. In the previous section we have shown that the choice of initial acceleration is unique in order that the solution is regular in the neighbourhood of $\varepsilon = 0$. It may be pointed out that the magnitude of the bounds are determined by the external field; hence our choice of initial acceleration should also be determined by the field, as in equations (2.4) and (2.9). In the absence of the field $M = 0$ and the only choice is $\dot{v}_\mu(0) = 0$ as stated above (see also equation (A.I.1)).

4. *Discussion*

It may be of interest to point out that according to Dirac (1938) the final acceleration should be prescribed in addition to the initial position and the initial velocity for unique solution. The physical content of the rule suggested in this paper and Dirac's prescription are the same if the final acceleration is bounded in time ($\tau > 0$). From the nature of the differential equation (2.1) it is quite clear that the solutions, which are singular for $\varepsilon = 0$, are unbounded for $\tau \to \infty$. This follows from the negative sign of $\varepsilon \ddot{v}_\mu$ in equation (2.1). On the other hand, if the sign was positive the general solutions would have been unbounded for $\tau \to -\infty$. This has also been noted by Dirac (1938). The prescription of the final acceleration may be very conveniently used only when the exact integrals are known. But the initial acceleration may be obtained successively to any order according to the present method and thereby the solution can be found to the required order. It may not be relevant to mention that this procedure is different from that of Bhabha (1946), according to which physical solutions are only those which are analytical in e. In this case the limiting motion, in the absence of the radiation reaction, is that of uniform velocity without any external field.

Appendix I

The General Solution in the Absence of the External Field

From equation (2.1), for $f_{\mu\nu} = 0$, it follows

$$\dot{v}_\mu \dot{v}^\mu = -w^2 e^{-2\tau/\varepsilon}, \qquad w^2 = -\dot{v}_\mu(0)\,\dot{v}^\mu(0) \tag{A.I.1}$$

$w^2 > 0$ as $\dot{v}_\mu$ is a space-like vector being orthogonal to v_μ. So that

$$\dot{v}_\mu - \varepsilon \ddot{v}_\mu + \varepsilon w^2 e^{2\tau/\varepsilon} v_\mu = 0 \tag{A.I.2}$$

Therefore,

$$v_\mu(\tau) = v_\mu(0) \cosh w\varepsilon(e^{\tau/\varepsilon} - 1) + \frac{\dot{v}_\mu(0)}{w} \sinh w\varepsilon(e^{\tau/\varepsilon} - 1) \tag{A.I.3}$$

Hence, in general, $\mathbf{v}(\tau)$ is not along a line but it is in the plane of containing $\mathbf{v}(0)$ and $\dot{\mathbf{v}}(0)$. $\mathbf{v}(\tau)$ is along a line only if $\mathbf{v}(0)$ and $\dot{\mathbf{v}}(0)$ are in the same direction. It is of interest to note that even when $\mathbf{v}(0) = 0$, $\mathbf{v}(\tau) \neq 0$ and it is along $\dot{\mathbf{v}}(0)$. The only initial value of $\dot{v}_\mu(0)$, for which $v_\mu(\tau)$ is bounded as $\tau \to \infty$, is $\dot{v}_\mu(0)\dot{v}^\mu(0) = 0$, i.e. $w = 0$ (from equation (A.I.1)). This also follows from our prescription in Section 2. The corresponding motion is with uniform velocity.

Appendix II

To obtain the extremum of

$$f_{\mu\nu} v^\nu u^\mu, \qquad f_{\mu\nu} + f_{\nu\mu} = 0$$

and

$$\left.\begin{matrix} \\ u_\mu u^\mu = -1, \qquad v_\mu v^\mu = 1, \qquad u_\mu v^\mu = 0 \end{matrix}\right\} \tag{A.II.1}$$

It is easy to find that the extremum is attained with u_μ and v_μ such that

$$f_{\mu\nu} u^\nu = -a v_\mu \qquad \text{and} \qquad f_{\mu\nu} v^\nu = a u_\mu \tag{A.II.2}$$

Hence,

$$f^\eta{}_\theta f^\theta{}_\phi w^\phi = a^2 w^\phi \qquad (w = u, v) \tag{A.II.3}$$

i.e. u_μ, v_μ, are eigenvectors of $\{f^\eta{}_\theta f^\theta{}_\phi\}$. In general, if $\mathbf{E}.\mathbf{H} \neq 0$, the eigenvalues of $f^\eta{}_\theta$ are λ, $-\lambda$, μ, and $-i\mu$ (λ and μ are real). Thus u_μ and v_μ are linear combinations of the eigenvectors corresponding to the eigenvalues λ and $-\lambda$. They are given by

$$4\lambda^2 = \{(\mathbf{H}.\mathbf{H} - \mathbf{E}.\mathbf{E})^2 + 4(\mathbf{E}.\mathbf{H})^2\}^{1/2} - \mathbf{H}.\mathbf{H} + \mathbf{E}.\mathbf{E} \tag{A.II.4}$$

Therefore,

$$-|\lambda| \leqslant f_{\mu\nu} u^\mu v^\nu \leqslant |\lambda| \tag{A.II.5}$$

In the case of $\mathbf{E}.\mathbf{H} = 0$, $\{f^\mu{}_\rho\}$ has a pair of eigenvectors with eigenvalues zero. The space of admissible (u,v) is the two-dimensional subspace orthogonal to the null-space of $\{f^\mu{}_\rho\}$ and the procedure for obtaining the extremum is similar to before. The problem of obtaining the initial acceleration is greatly simplified in this case, as two exact integrals of equation (2.1) exist when $\mathbf{E}.\mathbf{H} = 0$ (Sen Gupta, 1972).

References

Bhabha, H. J. (1946). *Physical Review*, **70**, 759.

Dirac, P. A. M. (1938). *Proceedings of the Royal Society*, **A169**, 148.

Sen Gupta, N. D. (1970). *Physics Letters*, **32A**, 103.

Sen Gupta, N. D. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 3, p. 179.

Sen Gupta, N. D. (1972a). *International Journal of Theoretical Physics*, Vol. 4, No. 5, p. 389.

Sen Gupta, N. D. (1972b). *Physical Review*, **D5**, 1546.